

*Extended theories of gravity and
fundamental physics:
Probing the Equivalence Principle at
Classical and Quantum Level*

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Summary

- Foundation: gravity and space-time
- Shortcomings in General Relativity
- Alternatives, way out and extensions
- The key role of conformal transformations
- Metric or connections?
- The role of Equivalence Principle
- Testing EP at classical and quantum level
- Conclusions

Foundation: gravity and space-time



Einstein worked hard at finding a theory of gravity based on the following requirements:

- principle of equivalence → Gravity and Inertia are indistinguishable; there exist observers in free fall (inertial motion)
- principle of relativity → Special Relativity holds pointwise; the structure of the spacetime is pointwise Minkowskian
- principle of general covariance → “democracy” in Physics
- principle of causality → all physical phenomena propagate respecting the light cones
- Riemann’s teachings about the link between matter and curvature →



Foundation: gravity and space-time



Mathematical consequences:

- principle of equivalence → Inertial motion = geodesic motion
- principle of relativity → the spacetime M is endowed with a Lorentzian metric g
- principle of general covariance → tensoriality
- principle of causality → light cones structure generated by the metric g
- Riemann's teachings about the link between matter and curvature → the gravitational field is described by $g \rightarrow 10$ equations
Riem(g) has 20 (independent) components: too many!
Ric(g) has 10 (independent) components: OK!

Foundation: gravity and space-time



The distribution of matter influences Gravity through 10 second order equations, nowadays called Einstein equations:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

A linear concomitant of the Riemann tensor, nowadays called the Einstein tensor, equals the stress-energy tensor that reflects the properties of matter.

They have a structure that suitably reduces to Newtonian equations in the “weak field limit.”

Foundation: gravity and space-time



Einstein was not so happy with the fact that the gravitational field is not the fundamental object, but just a by-product of the metric. Using a method invented few years before by **Attilio Palatini**, he realizes that one can obtain Einstein equations by working on a theory that depends on **two variables**, varied independently:

a **metric g** and a **linear connection Γ** assumed to be symmetric.

$$R \equiv R(g, \Gamma) = g^{\mu\nu} R_{\mu\nu}(\Gamma) \quad \mathcal{L}_{PE} = g^{\mu\nu} R_{\mu\nu}(\Gamma, \partial\Gamma)$$

There are **10 + 40** independent variables and the equations are:

$$R_{(\mu\nu)} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}$$

$$\nabla_{\alpha}^{\Gamma} (\sqrt{g} g^{\mu\nu}) = 0$$

Shortcomings in General Relativity

Is still g the fundamental object of Gravity?

Einstein tries to consider directly the connection as the fundamental object of Gravity, but he never completes the process of “dethronizing” g .

Shortcomings in General Relativity

But after all, what are the problems with GR?

GR is *simple*, beautiful.. but seems to be not self-consistent at all scales:

- cosmological constant Λ
- Inflation
- Dark Matter + Dark Energy
- Quantum Gravity problem
- Consistency of EP at classical and quantum level

Today observations say that there is way too few matter in the Universe! Thence the need, in order to save GR, for dark energy and dark matter:

$$G_{\mu\nu} = T_{\mu\nu} + T_{\mu\nu}^{dark}$$

Is there any way out to these shortcomings?

Alternatives, way out and extensions

A particular family of NLTG is that of $f(R)$ theories in metric formalism, in which the Hilbert Lagrangian is replaced by any non-linear density depending on R . GR is retrieved in (and only in) the particular case $f(R)=R$.

In these theories there is a second order part that resembles Einstein tensor (and reduces to it if and only if $f(R) = R$) and a fourth order “curvature part” (that reduces to zero if and only if $f(R) = R$):

$$f'(R(g)) R_{\mu\nu}(g) - \frac{1}{2} f(R(g)) g_{\mu\nu} - \nabla_{\mu} \nabla_{\nu} f'(R(g)) + g_{\mu\nu} \square f'(R(g)) = \kappa T_{\mu\nu}$$

Higher order Gravity (4th)!

Pushing the 4th order part to the r.h.s. lets interpret it as an “extra gravitational stress” $T_{\mu\nu}^{\text{curv}}$, much in the spirit of Riemann.

These theories can be recast in scalar-tensor form so the paradigm is that higher order terms can be dealt under the standard of scalar fields.

Alternatives, way out and extensions

From $f(R)$ theories GR is retrieved in (and only in) the particular case $f(R)=R$.

$$f'(R) = 1$$



$$f''(R) = 0$$



Second Order
Field
Equations



degenerate
theory

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 4\pi T_{\mu\nu}$$



Gravitational contribution



Matter contribution

The key role of conformal transformations

A conformal transformation is a transformation of a metric structure that does not change the angles but changes the scale.

In linear Euclidean Geometry this is just a dilatation.

In Riemannian Geometry it is a relation between two metrics that requires them to be proportional to each other by a conformal factor, assumed to be positive for obvious signature reasons.

$$\tilde{g} = \phi g \quad \sqrt{\tilde{g}} = \phi^2 \sqrt{g}$$

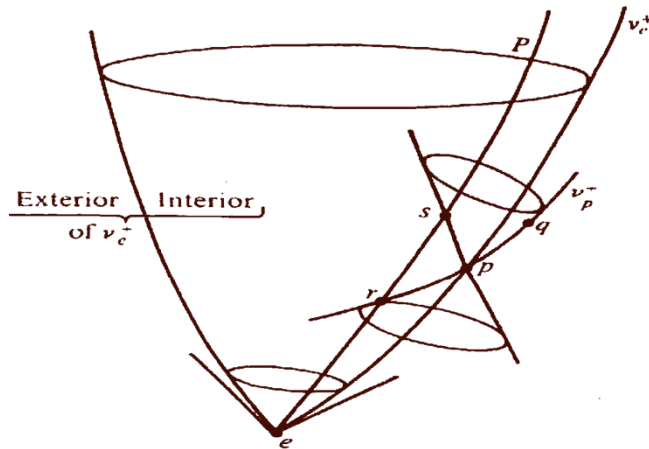
$$\tilde{R} = \phi^{-1} \left(R + (d-2)\Delta_g \phi + (d-1)(d-2)|\nabla \phi|_g^2 \right)$$

The key role of conformal transformations

The principle of causality determines a conformal structure in the spacetime, i.e. it selects not a metric but a whole family of conformally related metrics.

In fact the distributions of light cones of two conformally related (Lorentzian) metrics are necessarily the same, so that photons of the two metrics are the same (a photon is a particle that travels in such a way that its velocity lies in the light cone at each point of the trajectory of the particle).

Null geodesics are left unchanged by conformal transformations



The key role of conformal transformations

What happens to the Hilbert action if the metric undergoes a conformal transformation?

Or, in other words, is GR conformally invariant?

The answer is: of course not. If g is replaced by a conformally related metric, the action is multiplied by a factor and an additional term appears

Taking away a divergence (that does not affect field equations) it reduces to a conformal Hilbert action plus a second order action in the conformal factor ϕ .

In a sense the dynamics of g is driven by this factor, that obeys fourth order equations. The theory is not conformally invariant, since the leading equation that would impose conformal invariance has only particular solutions (e.g. $\phi = 0$).

$$A(\tilde{R}) = \int \tilde{R} \sqrt{\tilde{g}} = \int \phi (R + 2\Delta_g \phi + 6|\nabla \phi|_g^2) \sqrt{g}$$



$$\nabla_g \phi + 3|\nabla \phi|_g^2 = 0$$

Metric or connections?

Let's return to our Questions:

- Was Einstein right in assuming the metric g of the space-time as the fundamental object to describe Gravity?

Has conformal transformation a physical meaning?

Metric or connections?

When Einstein formulated GR, the only geometrical field he could use was a (Lorentzian) metric g , the structure that Gauss (1830) and Riemann (1856) introduced in surfaces and higher-dimensional manifolds to define curvature.

At that time he had no other choice.

In GR $\Gamma_{\mu\nu}^{\alpha} = \{\alpha\}_{\mu\nu}^g$ are not equations. They express a founding issue.

Assumption on space-time structure: there is a connection Γ ; this connection has no dynamics; it is in fact *a priori* the Levi-Civita connection of the metric g . Only g has dynamics. So the single object g determines at the same time the **causal structure** (light cones), the measurements (rods and clocks) and the **free fall** of test particles (geodesics). Spacetime is a couple (M, g) .

Even if it was clear to Einstein that Gravity induces “freely falling observers” and that the principle of equivalence selects in fact an object that cannot be a tensor - since it is capable of being “switched off” and set to zero at least in a point - he was obliged to choose it as being determined by the metric structure itself.

Metric or connections?

When in 1919 Levi-Civita introduced connections, Einstein had another choice. But he did not really take it. Why?

In Palatini's formalism a (symmetric) connection Γ and a metric g are given and varied independently. Spacetime is a triple (M, g, Γ) where the metric determines rods and clocks (i.e., it sets the fundamental measurements of spacetime) while Γ determines the free fall

The second equation tells us *a posteriori* that Γ is the Levi-Civita's connection of g . The first equation is then turned into the standard Einstein's equation. That is probably why Einstein followed to consider the metric as the fundamental object of Gravity

.. But this coincidence (between Γ and the Levi-Civita's connection of g) is entirely due to the particular Lagrangian considered by Einstein, which is the *simplest*... but not the only one possible!

Metric or connections?

Furthermore Einstein did not recognize that Palatini method privileges the affine structure towards the metric structure

In fact the Einstein-Palatini Lagrangian contains only derivatives of Γ , that is the real dynamical field. The metric g has no dynamics since it enters the Lagrangian as a “Lagrange multiplier”

The metric g gains DYNAMICS from Γ

The dynamics of Γ tells us that a sort of Einstein’s equation holds for the Ricci’s tensor of Γ

This dynamics is obtained by varying the Lagrangian with respect to the metric. These are 10 equations. Other 40 equations come out when varying the Lagrangian with respect to the connection Γ . These additional equations govern the form of Γ and impose it to be the Levi-Civita connection of the metric. The first equation then transforms into Einstein Equations.

In Palatini’s formalism $\Gamma_{\mu\nu}^{\alpha} = \{\alpha_{\mu\nu}\}_g$ are now equations.

The fact that Γ is the Levi-Civita connection of g is no longer an assumption but becomes an outcome of field equations!

Metric or connections?

Among the different Theories of Gravitation, we really should prefer the simplest (in the sense of the one with the simplest Lagrangian)?

Metric or connections?

The universality properties discovered for non-linear theories of Gravitation, written under the Palatini form, tell us that the true dynamical field is Γ and not the metric g .

The metric g is no longer a Lagrange multiplier, but still has no dynamics since it enters algebraically the Lagrangian. However g gains dynamics from the dynamics of the connection Γ .

The connection is the Gravitational Field and, as such, it is the fundamental field in the Lagrangian. The metric g enters the Lagrangian with an “ancillary role.”

It reflects the fundamental need we have to define length and distances, as well as areas and volumes. It defines rods & clocks, that we use to make experiments. It defines also the causal structure of spacetime.

But it has no dynamical role.

There is no reason whatsoever to assume g to be the potential for Γ .

Nor that it has to be a true field just because it appears in the action!

Metric or connections?

Newtonian and Galilean Physics are based on the Geometry of Euclidean space.

Both the affine and the metric structure enter the game.

These two structures are separated, one is not obtained from the other.

The structure of Newtonian Physics is a subtle mix-up of both structures..... but the principle of inertia selects the family of parallel lines, that become “more fundamental” than the metric structure!

The Physics of Gravitation is based on the Geometry of a curved 4D spacetime. An affine and a metric structure enter the game.

These two structures are separated. One is not obtained from the other. The structure of Gravitational Physics is a subtle mix-up of both structures.

... but the Principle of Equivalence selects the family of geodesics of Γ , that become “more fundamental” than the metric structure of g ! The principle of equivalence selects the true dynamical field; rods & clocks follow up.

In summary: We must suitably discriminate among different theories of gravity!!!

$$G_{\mu\nu} \Rightarrow \tilde{G}_{\mu\nu} \quad \text{---} \quad G_{\mu\nu} = (8\pi G) T_{\mu\nu} \quad \text{---} \quad T_{\mu\nu} \Rightarrow \tilde{T}_{\mu\nu}$$

Extended theories of Gravity

WHY?

- QFT on curved spacetimes
- String/M-theory corrections
- Brane-world models



$$R, R^{\mu\nu} R_{\mu\nu}, R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}, R \square^l R,$$

Curvature invariants or scalar fields have to be taken into account

Dark Energy and Λ

- Cosmological constant (Λ)
- Time varying Λ
- Scalar field theories
- Phantom fields
- Phenomenological Theories
- Exotic matter

The role of Equivalence Principle

We need to investigate the EP:

- discriminating among theories of gravity
- its validity at classical and quantum level
- investigating geodesic and causal structure

The role of Equivalence Principle

Einstein Equivalence Principle states:

- Weak Equivalence Principle is valid;
- the outcome of any local non-gravitational test experiment is independent of velocity of free-falling apparatus;
- the outcome of any local non-gravitational test experiment is independent of where and when in the Universe it is performed.

The role of Equivalence Principle

One defines as “local non-gravitational experiment” an experiment performed in a small-size freely falling laboratory

One gets that the gravitational interaction depends on the curvature of space-time, i.e. the postulates of any metric theory of gravity have to be satisfied

- space-time is endowed with a metric $g_{\mu\nu}$;
- the world lines of test bodies are geodesics of the metric;
- in local freely falling frames, called local Lorentz frames, the non-gravitational laws of physics are those of Special Relativity;

The role of Equivalence Principle

One of the predictions of this principle is the gravitational red-shift, experimentally verified by Pound and Rebka in 1960

Gravitational interactions are excluded from WEP and Einstein EP

In order to classify alternative theories of gravity, the Gravitational WEP and the Strong Equivalence Principle (SEP) has to be introduced

The role of Equivalence Principle

The SEP extends the Einstein EP by including all the laws of physics in its terms:

- WEP is valid for self-gravitating bodies as well as for test bodies (Gravitational Weak Equivalence Principle);
- the outcome of any local test experiment is independent of the velocity of the free-falling apparatus;
- the outcome of any local test experiment is independent of where and when in the Universe it is performed.

The SEP contains the Einstein Equivalence Principle, when gravitational forces are neglected

The role of Equivalence Principle

Many authors claim that the only theory coherent with the SEP is GR

An extremely important issue is related to the consistency of EP with respect to the Quantum Mechanics.

Some phenomena, like **neutrino oscillations** could violate it if induced by the gravitational field.

GR is not the only theory of gravitation and, several alternative theories of gravity have been investigated from the 60's, considering the space-time to be "special relativistic" at a background level and treating gravitation as a Lorentz-invariant field on the background

The role of Equivalence Principle

Two different classes of experiments can be considered:

- the first ones testing the foundations of gravitation theory (among them the EP)
- the second one testing the metric theories of gravity where space-time is endowed with a metric tensor and where the Einstein EP is valid.



For several fundamental reasons extra fields might be necessary to describe the gravitation, e.g. scalar fields or higher-order corrections in curvature invariants.

The role of Equivalence Principle

Two sets of equations can be distinguished

- The first ones couple the gravitational fields to the non-gravitational contents of the Universe, i.e. the matter distribution, the electromagnetic fields, etc...
- The second set of equations gives the evolution of non-gravitational fields.

Within the framework of metric theories, these laws depend only on the metric: this is a consequence of the EEP and the so-called "minimal coupling".

Several theories are characterized by the fact that a scalar field (or more than one scalar field) is coupled or not to gravity and ordinary matter

There are several reasons to introduce a scalar field:

- * Scalar fields are unavoidable for theories aimed to unify gravity with the other fundamental forces: e.g. Superstring, Supergravity (SUGRA), M-theories.
- * Scalar fields appear both in particle physics and cosmology:
 - the Higgs boson in the Standard Model
 - the dilaton entering the supermultiplet of higher dimensional gravity
 - the super-partner of spin $\frac{1}{2}$ in SUGRA.

• The introduction of a scalar field gives rise typically to a possible “violation” of the Einstein Equivalence Principle (EEP).



In order to distinguish competing theories, a possibility is related to the so-called “fifth force” approach. For example, the case of $f(R)$ -gravity:

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[f(R) + \mathcal{X} \mathcal{L}_m \right], \quad \mathcal{X} = \frac{16\pi G}{c^4}$$

The variation with respect to the metric tensor gives

$$f' R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - f'_{;\mu\nu} + g_{\mu\nu} \square f' = \frac{\mathcal{X}}{2} T_{\mu\nu}$$

$$3\square f' + f' R - 2f = \frac{\mathcal{X}}{2} T \quad \text{Trace equation}$$

$$f(R) = \sum_n \frac{f^n(R_0)}{n!} (R - R_0)^n \simeq f_0 + f_1 R + f_2 R^2 + f_3 R^3 + \dots$$

In the Newtonian limit,
let us consider the perturbation of the metric
with respect to the Minkowski background

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

The metric entries can be developed as

$$\left\{ \begin{array}{l} g_{tt}(t, r) \simeq 1 + g_{tt}^{(2)}(t, r) + g_{tt}^{(4)}(t, r) \\ g_{rr}(t, r) \simeq -1 + g_{rr}^{(2)}(t, r) \\ g_{\theta\theta}(t, r) = -r^2 \\ g_{\phi\phi}(t, r) = -r^2 \sin^2 \theta \end{array} \right. ,$$

As general solution:

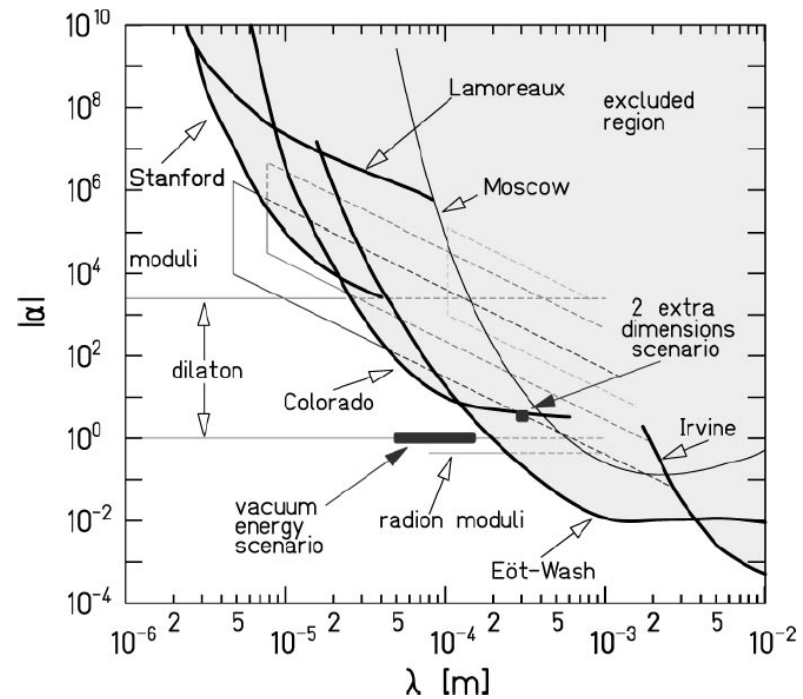
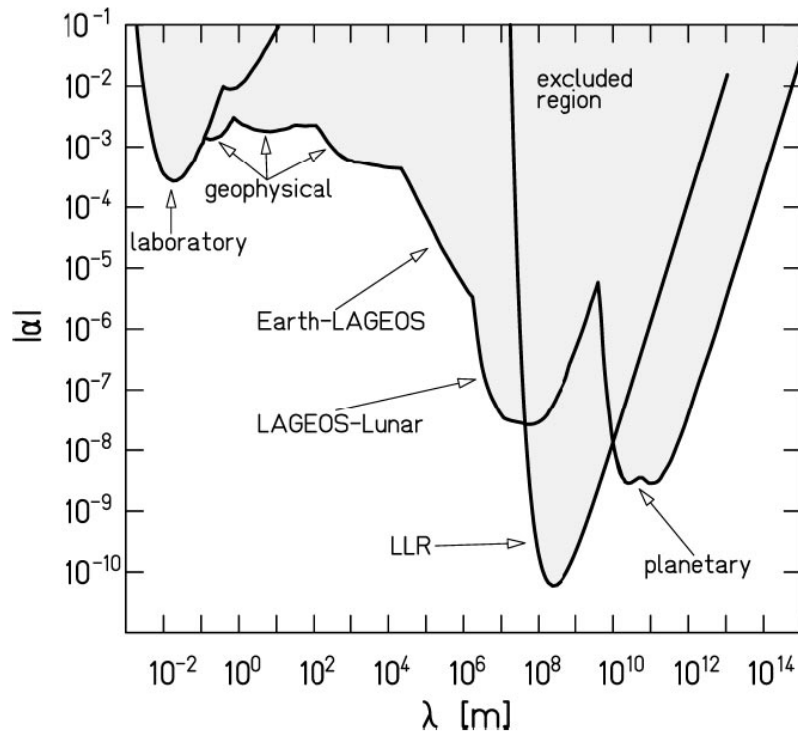
$$\Phi_{grav} = - \left(\frac{GM}{f_1 r} + \frac{\delta_1(t) e^{-r\sqrt{-\xi}}}{6\xi r} \right)$$

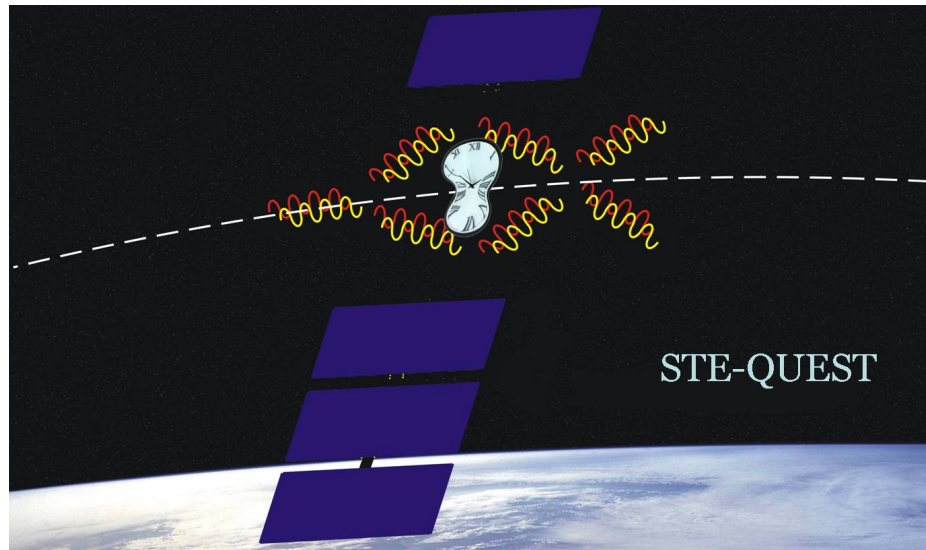
Fifth force $V(r) = -G \frac{m_1 m_2}{r} [1 + \alpha e^{-r/\lambda}]$

α is a dimensionless strength parameter

λ is a length scale or range

Experimental bounds





- In conclusion,
the STE-QUEST Experiment could allow:
- to set and refine the bounds on space parameters
 - to discriminate among competing theories
 - fifth force
 - to test EP and SEP at quantum level
 -Wait and see...